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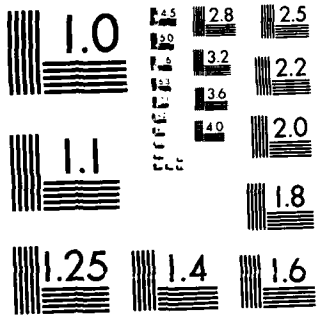

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A GRAPH WITH E EDGES HAS PAGENUMBER $O(\sqrt{E \log E})$

Seth M. Malitz

The paper presents the results,

Abstract

A book embedding of a graph consists of a linear ordering of the vertices along the spine of a book and an embedding of its edges on the pages so that no two edges on a given page intersect. The minimum number of pages in which a graph can be embedded is its pagenumber. ~~In this paper, we present the following results:~~

1. An $O(\sqrt{E \log E})$ upper bound on the pagenumber of an arbitrary graph with E edges and any number n of nodes. In the case $E > 4n$, this is a dramatic improvement over the $O(E)$ bound given in [HI]. Furthermore, this result is nearly tight since a clique with E edges has pagenumber $\Omega(\sqrt{E})$.

2. With high probability, a random graph with roughly E edges has pagenumber $O(\sqrt{E})$.

3. An $O(\sqrt{d \log d}\, n^{1/2})$ upper bound on the pagenumber of any d-regular graph. For large d, this is a marked improvement over the $O(dn^{1/2})$ upper bound given in [CLR].

4. For every $d > 2$ and all large n, there are n-vertex d-regular graphs with pagenumber $\Omega(\sqrt{d}n^{1/2-1/d})$. This significantly improves the $\Omega(n^{1/2-1/d}/\log^2 n)$ lower bound given in [CLR].

(Keywords: upper bounds; lower bounds; probab[illegible]),

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A Graph with E Edges has Pagenumber $O(\sqrt{E \log E})$

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November 2, 1987

Abstract

A book embedding of a graph consists of a linear ordering of the vertices along the spine of a book and an embedding of its edges on the pages so that no two edges on a given page intersect. The minimum number of pages in which a graph can be embedded is its pagenumber. In this paper, we present the following results:

1. An $O(\sqrt{E \log E})$upper bound on the pagenumber of an arbitrary graph with E edges and any number n of nodes. In the case $E > 4n$, this is a dramatic improvement over the $O(E)$ bound given in [HI] . Furthermore, this result is nearly tight since a clique with E edges has pagenumber $\Omega(\sqrt{E})$.
2. With high probability, a random graph with roughly E edges has pagenumber $O(\sqrt{E})$.
3. An $O(\sqrt{d \log d} n^{1/2})$ upper bound on the pagenumber of any d-regular graph. For large d, this is a marked improvement over the $O(dn^{1/2})$ upper bound given in [CLR].
4. For every $d > 2$ and all large n, there are n-vertex d-regular graphs with pagenumber $\Omega(\sqrt{d} n^{1/2-1/d})$. This significantly improves the $\Omega(n^{1/2-1/d}/\log^2 n)$ lower bound given in [CLR].

1 Introduction

A book embedding of a graph orders the vertices of the graph on the spine of a book and embeds each edge in a page so that on any page, no two edges cross. The minimum number of pages in which a graph can be embedded is its pagenumber.

Recent interest in book embeddings has been motivated by fault-tolerant VLSI design ([R],[CLR]), and by complexity theory ([GKS],[K],[PPST]). With the Diogenes approach to the design of fault-tolerant processor arrays [R], we do the following. An array of processors has to be interconnected to form a desired pattern. The processors are arranged in a "conceptual" line. There are "bundles" of wires running alongside the line. The faulty elements are bypassed and the healthy ones are interconnected through the bundles. If bundles function as stacks, then the minimum number of bundles required to realize an interconnection graph is equal to its pagenumber. Thus a book embedding with a small number of pages corresponds to reduced hardware. Since the Diogenes methodology assumes as input an arbitrary interconnection graph, it is desirable to have good bounds on pagenumbers for large classes of graph. Unfortunately, even if an ordering of the nodes

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is specified beforehand, the problem of determining an optimal edge embedding for general graphs is NP-complete [GJMP].

Some other notable results concerning pagenumbers are the following. Bernhart and Kainen [BK] proved that graphs with pagenumber 3, could have arbitrarily large genus. In the other direction, they conjectured that graphs of a fixed genus could require an unbounded number of pages. In the case of genus 0, i.e. planar graphs, this conjecture was disproved Buss and Shor [BS] who showed how to embed a planar graph in 9 pages. Later Heath [H] improved that to 7 pages and developed book embedding algorithms for special classes of planar graphs. Finally Yannakakis [Y], building on the approach in [H] , gave an algorithm to embed planar graphs in 4 pages and also constructed a planar graph requiring 4 pages. The general conjecture for arbitrary fixed genus was recently disproved by Heath and Istrail [HI] who provide an algorithm to embed a genus g graph in $O(g)$ pages. Finally, Chung, Leighton and Rosenberg [CLR] establish many interesting results concerning pagenumbers. They provide optimal book embeddings for a variety of important networks, supply nontrivial upper and lower bounds on the pagenumbers of d-regular graphs, relate pagenumber to bisection width, and obtain an algorithm to embed 3-regular graphs in $O(n^{1/2})$ pages.

Our paper is motivated by the results in [HI] and [CLR]. Heath and Istrail [HI] showed that a graph G of genus g can be embedded in $O(g)$ pages and conjectured that in fact $O(\sqrt{g})$ pages would suffice. If G has n nodes and E edges, and $E \leq 3n$, then g could be much less than E. In this case, it is a much stronger statement to say G can be embedded in $O(\sqrt{g})$ pages than it is to say G can be embedded in $O(\sqrt{E})$ pages. If, on the other hand, $E > 4n$, then $g = \Theta(E)$ and thus to say G can be embedded in $O(\sqrt{g})$ pages is equivalent to saying G can be embedded in $O(\sqrt{E})$ pages. This puts our result of Section 2 in the proper perspective. There we show that any graph with E edges can be embedded in $O(\sqrt{E \log E})$ pages, and this is nearly tight since the clique with E edges has pagenumber $\Omega(\sqrt{E})$. In Section 3, we show that a random graph with roughly E edges has pagenumber $O(\sqrt{E})$ with high probability. Section 4 provides an $O(\sqrt{d \log d} n^{1/2})$ upper bound on the pagenumber of d-regular graphs which for large d, markedly improves the $O(dn^{1/2})$ bound given in [CLR]. Finally, in Section 5, we argue that for $d > 2$ and all sufficiently large n, some d-regular graph requires $\Omega(\sqrt{d} n^{1/2-1/d})$ pages. This significantly improves the $\Omega(n^{1/2-1/d}/\log^2 n)$ bound of [CLR].

2 The $O(\sqrt{E \log E})$ Upper Bound

Let G be a graph with n nodes and E edges, and assume $E > n$. Why might one guess that G could be embedded in close to $O(\sqrt{E})$ pages? There is a two part answer. First, the complete graph with E edges has pagenumber $O(\sqrt{E})$. And second, by picking a random partition of the nodes into $\sqrt{E}$ "supernodes" of size $n/\sqrt{E} < \sqrt{E}$, one expects a rather uniform distribution of the edges among supernode pairs. In effect, we can almost simulate the complete graph with E edges, and thus almost obtain an $O(\sqrt{E})$-page embedding with any linear order that proceeds supernode by supernode.

In fact, this simple intuition fails us somewhat, but it does embody our basic approach to the problem. Various modifications of this idea will take us a long way.

To gain a little more intuition for the problem, we make a few observations.

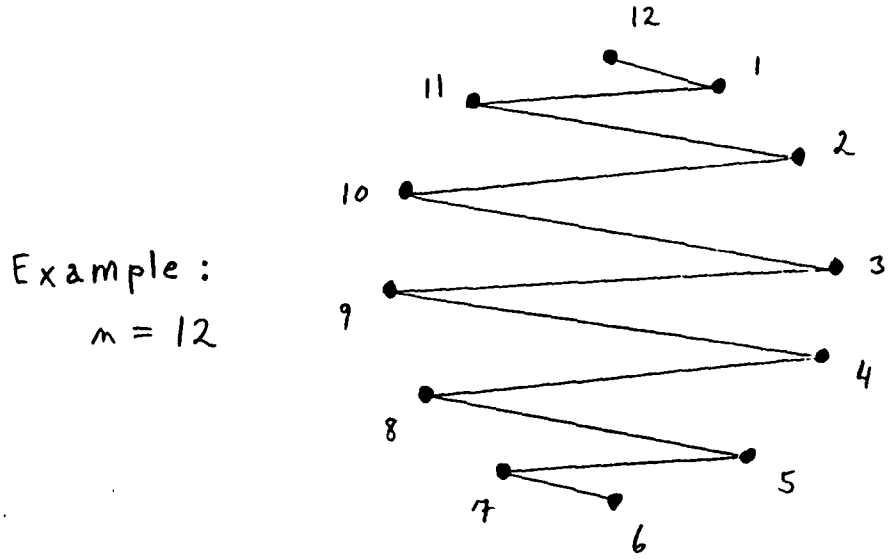

Figure 1:

Lemma 2.1 [CLR]. Assume n is even. The complete graph on n vertices, K_n, has pagenumber $n/2$.

Proof. The lower bound on pagenumber is argued as follows. Lay the vertices of K_n out on a line; call the vertices $1, \ldots, n$ in left-to-right order. Consider the following set of edges $\{(1, 1 + n/2), (2, 2 + n/2), \ldots, (n/2, n)\}$, and observe that no pair of edges from this collection can be placed on the same page without intersecting. Hence this embedding requires $n/2$ pages. Since all embeddings of K_n are isomorphic, the lower bound on pagenumber follows.

To see the upper bound, consider the following way to lay out K_n. Place vertices $1, \ldots, n$ evenly spaced on a circle and look at the path P_n which orginates at vertex 1 in Figure 1.

If we take the union of P_n and $(n/2) - 1$ adjacent clockwise rotates of P_n, we get K_n. In fact,one can verify that each edge of K_n appears in exactly one of the $n/2$ rotates of P_n. Since each rotate of P_n is outerplanar, we can place it on its own page, thereby obtaining an $n/2$ page embedding of K_n. •

Observation. Let G be an arbitrary graph, and suppose it is possible to partition the nodes of G into t supernodes all of size at most s in such a way that at most c edges span any pair of supernodes. Then by repeated application of Lemma 2.1, G can be embedded in at most $(s/2) + ct$ pages.

To see this, first linearly order the vertices of G in any fashion that proceeds supernode by supernode. Fix for a moment some supernode U. The edges with both endpoints in U can be embedded in $s/2$ pages by Lemma 2.1. Since our linear order proceeds supernode by supernode, the edges within one supernode can never intersect the edges within another. Consequently, all edges that have both endpoints in the same supernode can be embedded in $s/2$ pages.

Now we must focus on the "superedges", the edges that span pairs of supernodes. Assign a color between 1 and c to each superedge in such a way that no two edges spanning the same pair of supernodes get the same color. Consider all edges of color i. By using an embedding strategy analagous to Lemma 2.1, the edges of color i can be embedded in at most $2(t/2) = t$ pages. The factor of 2 arises for the following reason. Look at a set of i-colored edges that form what is analagous to P_t in Lemma 2.1. Call this graph P_t'. Two edges of P_t' incident to the same supernode U may not have the same endpoint in U and hence the linear order of the vertices may force these edges to intersect. However, two edges of P_t' that are not incident to the same supernode never

intersect. Thus we can embed P_i' in 2 pages, placing every other edge on one page and the remaining edges on the second page. If we do this now for all the colors 1 through c, we obtain an embedding of all superedges in at most ct pages.

Thus $(s/2) + ct$ pages suffice to embed all the edges. •

A set of edges in a graph G is said to be **independent** if no pair of edges in the set share an endpoint. Such a set will also be called a **matching**. A **star** is any complete bipartite graph $K_{1,r}$.

Lemma 2.2. Let G be an arbitrary graph and suppose the nodes of G can be partitioned into t supernodes of size at most s in such a way that at most c *independent* edges span any pair of supernodes. Then G can be embedded in at most $(s/2) + ct$ pages.

Proof. Look at the bipartite graph B induced by a pair of supernodes. We know, by assumption, that B has no matching of size $> c$. Recall the well-known fact that | maximum matching | = | minimum cover | in a bipartite graph, decompose B into a set of at most c edge-disjoint stars and assign each star a different color between 1 and c. Do this for all pairs of supernodes. Now consider any linear ordering of the vertices of G that proceeds supernode by supernode. If we view each star now as being a "superedge", we can use the exact same argument as in the observation to show that G can be embedded in $(s/2) + ct$ pages. •

Finally, assume we have an arbitrary graph G with n nodes and E edges. With the aid of the next two simple lemmas, we can establish that G can be embedded in $O(\sqrt{E \log E})$ pages.

Randomly assign a color in $\{1, \ldots, \sqrt{2E/p}\}$ (p to be defined shortly) to each node of G uniformly and independently, and group nodes of the same color into supernodes. Let A_k^{ij} $i, j \in \{1, \ldots, \sqrt{2E/p}\}$ $k \geq 0$ be the event that k independant edges span supernodes i and j. For k fixed, the indicator random variables for the A_k^{ij} are identically distributed, so superscripts may be dropped.

Lemma 2.3.

$$\Pr[\bigvee_{\{i,j\}, i \neq j} A_k^{ij}] < \frac{E}{p} \frac{p^k}{k!}.$$

Proof. Fix a set S of k independent edges in G. The probability that all edges in S have one endpoint colored i and the other colored j is $2^k(1/\sqrt{2E/p})^{2k} = p^k/E^k$. The total number of such S is $\binom{E}{k} < E^k/k!$. Thus $\Pr[A_k] < (E^k/k!)(p^k/E^k) = p^k/k!$ and the result follows. •

Lemma 2.4. Let $a > 1$. The probability that any supernode exceeds cardinality $an/\sqrt{2E/p}$ is $< n^2(e/a)^{n/\sqrt{2E/p}}$.

Proof. Standard estimates of the binomial distribution. •

Theorem 2.1. Let G be an arbitrary graph with n nodes and E edges. Then with high probability, a random linear ordering of the nodes of G is compatible with an $O(\sqrt{E \log E})$-page book embedding of G.

Proof. Let $p = (b/2e) \log E$ and $k = b \log E$. Then $\Pr[\bigvee_{\{i,j\}, i \neq j} A_k^{ij}] < E/E^b$. Let $a = 2e$ and $b = 2$. By Lemmas 2.3 and 2.4, with high probability no supernode has more than $(2e)n/\sqrt{4eE/\log E} < \sqrt{2e}\sqrt{E \log E}$ nodes, and no pair of supernodes is spanned by more than $2 \log E$ independent edges. Now apply Lemma 2.2 with $t = \sqrt{2E/p} = \sqrt{4eE/\log E}$, $s = \sqrt{2e}\sqrt{E \log E}$ and $c = 2 \log E$. With high probability a random node coloring is compatible with an embedding of G in $(s/2) + ct = (\sqrt{e/2} + 4\sqrt{e})\sqrt{E \log E}$ pages. •

3 An $O(\sqrt{E})$ Upper Bound for Random Graphs

We are exceedingly close to a proof that any graph with E edges can be embedded in $O(\sqrt{E})$ pages, and hope to include it in our final paper. What has been a bit of a hindrance is the following dependency problem. Let $p_1, \ldots, p_t$ denote arbitrary pairs of supernodes, and let $k_1, \ldots, k_t$ be a sequence of integers all ≥ 0. It is not typically the case that the events $A_{k_1}^{p_1}, \ldots, A_{k_t}^{p_t}$ are independent. If they were independent or if $\Pr[A_{k_1}^{p_1} \wedge \cdots \wedge A_{k_t}^{p_t}] \leq \Pr[A_{k_1}^{p_1}] \cdots \Pr[A_{k_t}^{p_t}]$, then we would have a proof that any graph with E edges has pagenumber $O(\sqrt{E})$.

One way of getting around this dependency problem, is to consider random graphs. In particular, we can show that with high probability, a random graph on n nodes obtained by throwing in each edge independently with probability E/n^2, has an $O(\sqrt{E})$-page book embedding.To prove this, we essentially apply the ideas of Section 2 in an inductive manner, defining a process that first fixes the order of the vertices and then proceeds to embed the edges of G in phases.

Again G is a random graph on n nodes obtained by throwing in edges independently with probability E/n^2. Fix a sufficiently large *constant* d. We will obtain d later.

PHASE 0: Fix a linear ordering of the vertices of G and partition the ordering into $\sqrt{E}$ consecutive blocks of size $n/\sqrt{E}$. We will refer to these blocks as "0-supernodes." Decompose the bipartite graph determined by each pair of 0-supernodes into a minimal number of stars. Call these stars "base-stars." Remove the edges of d base-stars from between each pair of 0-supernodes. By Lemma 2.2, these edges can be embedded in $d\sqrt{E}$ pages.

PHASE $i+1$: We have $\sqrt{E}/2^i$ i-supernodes. Pair neighboring i-supernodes to form a set of $\sqrt{E}/2^{i+1}$ $(i+1)$-supernodes. From between each pair of $(i+1)$-supernodes, remove the edges in d base-stars. By Lemma 2.2, these edges can be embedded in $d\sqrt{E}/2^{i+1}$ pages.

Let ${}_sA_k^{ij}$ where $i, j \in \{1, \ldots, \sqrt{E}/2^s\}$, $s, k \geq 0$ be the event that $\geq k$ base-stars span s-supernodes i and j after phase s. For k and s fixed, the indicator random variables for the events ${}_sA_k^{ij}$ are identically distibuted, so we may drop subscripts.

Lemma 3.1. $\Pr[{}_0A_k] < 1/k!$.

Proof. Omitted but very easy. •

Hence Pr[There are i, j such that ${}_0A_k^{ij}$ holds]$< E/k!$. Thus with high probability, no pair of 0-supernodes is spanned by a matching of size $> \log E$.

Suppose that with high probability, after $\lambda = (1/2)\log\sqrt{E}$ phases there are no edges left spanning any pair of λ-supernodes. Each λ-supernode contains $\sqrt[4]{E}$ 0-supernodes and there are $\sqrt[4]{E}$ λ-supernodes. By Lemma 2.2 and the above remark, with high probability all edges wholly within λ-supernodes can be embedded in $(n/2\sqrt{E}) + (\log E)\sqrt[4]{E} = O(\sqrt{E})$ pages. Thus with high probability, all edges of G can be embedded in

$$O(\sqrt{E}) + d\sqrt{E}/2 + d\sqrt{E}/4 + \cdots + d\sqrt{E}/2^{\lambda} = O(\sqrt{E})$$

pages. So the idea now is to show that $\Pr[{}_\lambda A_1] \ll 1/\sqrt[4]{E^2} = 1/\sqrt{E}$. Here is where generating functions can be quite useful.

Consider the coefficient on the x^k term in the series expansion of e^x. Lemma 3.1 says this an upper bound on $\Pr[{}_0A_k]$. Not surprisingly, we can manipulate the series e^x in phases, so that after phase s we obtain an upper bound on $\Pr[{}_sA_k]$.

PHASE 0: $g_0(x) \Leftarrow 1 + (e^x - \text{terms } x^0 \text{ thru } x^d)/x^d$.

PHASE $i+1$: $g_{i+1}(x) \Leftarrow 1 + ([g_i(x)]^4 - \text{terms } x^0 \text{ thru } x^d \text{ of } [g_i(x)]^4)/x^d$.

Lemma 3.2. $\Pr[_sA_k]$ <coefficient on x^k term of $g_s(x)$.

Proof. Omitted but easy.

With a bit of analysis, one can nicely upper bound the coefficients of $g_s(x)$. It turns out that if d is chosen sufficiently large, the coefficient on the x term of $g_\lambda(x)$ is $\ll 1/\sqrt{E}$. Thus we have

Theorem3.1. With high probability, a random graph on n nodes obtained by throwing in each edge independently with probability E/n^2, has pagenumber $O(\sqrt{E})$.

4 An $O(\sqrt{d \log d} n^{1/2})$ Upper Bound for d-regular Graphs

Let G be a d-regular graph on n vertices. Since G has $E = dn/2$ edges, the result of Section 2 gives an upper bound of $O(\sqrt{(dn)\log(dn)})$ on the pagenumber of G. This result can be improved to $O(\sqrt{d \log d} n^{1/2})$ using the Lovasz Local Lemma, which we now recall.

Let $\{F_1, \ldots, F_t\}$ be a set of events. A graph H on the vertices $[t]$ is called a **dependency graph** for $F_1, \ldots, F_t$ if for all i, F_i is mutually independent of all F_j with $\{i,j\} \notin H$. (That is, F_i is independent of any Boolean function of these F_j).

Lovasz Local Lemma. Let $F_1, \ldots, F_t$ be events with dependency graph H such that

$$\Pr[F_i] \leq q \text{ all } i,$$

$$\deg(i) \leq \delta \text{ all } i$$

and

$$4\delta q < 1.$$

Then $\Pr[\bigwedge \overline{F}_i] > 0$. •

As in Section 2, randomly assign a color in $\{1, \ldots, \sqrt{2E/p}\}$ (p to be defined shortly) to each node of G uniformly and independently, and group nodes of the same color into supernodes.

Fix $k \geq 0$. For each k-matching M, let F_M be the event that M spans some pair of supernodes. The dependency graph H is on the collection of all k-matchings. The degree of any node in H is bounded above by

$$\delta = 2dk\binom{E}{k-1} < 2dkE^{k-1}/(k-1)!\,.$$

All the F_M are equaly likely and

$$q = \Pr[F_M] < \binom{\sqrt{2}E/p}{2} 2^k (1/\sqrt{2E/p})^{2k} < p^{k-1}/E^{k-1}.$$

If we pick k so large that $4\delta q < 1$, then we can apply the Lovasz Local Lemma. We have

$$4\delta q < 4 \cdot 2dk \frac{E^{k-1}}{(k-1)!}\frac{p^{k-1}}{E^{k-1}} = 8dk\frac{p^{k-1}}{(k-1)!} \approx d\frac{p^k}{k!}$$

which is less than 1 if $p = (1/2e)\log d$ and $k = \log d$. Thus by the Lovasz Local Lemma, the probability that no pair of supernodes is spanned by more than $k = \log d$ independent edges

$= \Pr[\bigwedge \overline{F}_M] > 0$. By Lemmas 2.2 and 2.4, this shows an $O(\sqrt{d \log d} n^{1/2})$ upper bound on the pagenumber of G. We have just shown

Theorem 4.1. If G is a d-regular graph on n nodes, then G can be embedded in $O(\sqrt{d \log d} n^{1/2})$ pages.)

5 An $\Omega(\sqrt{d} n^{1/2-1/d})$ Lower Bound for d-regular Graphs

Theorem 5.1. For $d > 2$ and all sufficiently large n, a random d-regular graph on n nodes has pagenumber $\Omega(\sqrt{d} n^{1/2-1/d})$ with high probability.

Proof Sketch. The main idea here is to show that with high probability, a random d-regular graph on n nodes (thus $E = dn/2$ edges) has the following property. No matter how the nodes are clustered into $\sqrt{E}$ supernodes of size $n/\sqrt{E}$, a fraction $\alpha(n,d)$ of the $\binom{\sqrt{E}}{2}$ supernode pairs have a spanning edge. To do this, one basically counts the number of d-regular graphs that don't have this property. Taking $\alpha(n,d) = O(n^{-1/d})$, this is a tiny fraction of d-regular graphs .

Now consider a d-regular graph G that does have the above property, and linearly order its vertices in any fashion. Partition the linear order into $\sqrt{E}$ consecutive blocks of size $n/\sqrt{E}$. Look at those pairs of blocks that are spanned by at least one edge, and for each such pair remove exactly one spanning edge. Let S be the set of removed edges. By assumption, $|S| > \alpha(n,d)\binom{\sqrt{E}}{2} \approx \alpha(n,d)E/2$. Since no more than $3\sqrt{E}$ of these edges can appear on the same page (a planar graph on t nodes has less than $3t$ edges), $\Omega(\alpha(n,d)\sqrt{E})$ pages will be required to embed the edges in S. Thus any book embedding of G requires $\Omega(\alpha(n,d)\sqrt{E})$ pages. •

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